Complete Axiomatization of Discrete-Measure Almost-Everywhere Quantification

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Abstract

Following recent developments in the topic of generalized quantifiers, and also having in mind applications in the areas of security and artificial intelligence, a conservative enrichment of (two-sorted) first-order logic with almost-everywhere quantification is proposed. The completeness of the axiomatization against the measure-theoretic semantics is carried out using a variant of the Lindenbaum–Henkin technique. The independence of the axioms is analyzed, and the almost-everywhere quantifier is compared with related notions of generalized quantification.

Keywords: generalized quantification, almost-everywhere logic, probabilistic logic, measure-theoretic semantics, complete axiomatization.

1 Introduction

The study of generalized quantifiers [7, 17, 14, 18, 20] has attracted attention in the last decades, mainly motivated by applications, such as in natural language [5], artificial intelligence [19, 12, 15, 10], and philosophy [21]. Applications in security suggest adopting a probabilistic interpretation of "for almost all" of the type considered in [12]. Such a type of quantification is also studied in [3, 6] but in the more general setting of a measure-theoretic semantics.

An important trend in the area of kleistic logic[1] is directed at developing formal calculi for reasoning about the probabilistic universe of security protocols, for instance in the context of encryption [2, 1, 16, 4, 9], but with no linguistic constructs denoting probabilities: these only appear at the semantic level.

Having in mind such applications in security, our aim was to develop a purely qualitative extension of first-order logic (FOL) with a quantifier Æ corresponding to the measure-theoretic notion of "almost everywhere". By purely qualitative we mean that there should be no language constructions denoting measure values. The key idea was to endow each first-order structure with a measure over some σ-algebra of subsets of the domain. This semantic approach had already been pursued to some extent in [3, 6], and

[1]Kleistic logic is the logic of security, from the Greek *kleisis*.

also in [12, 10]. However, the former allows only one almost-everywhere quantification applied to a FOL implication and does not provide a calculus, while the latter includes terms denoting probabilities or measures in the language.

The resulting logic FOL+AE, as described in Section 2, does not enjoy the downward Lowenheim–Skölem property, which strongly suggests that it will not be axiomatizable. In Section 3 we overcome this difficulty by adding quantification over unary predicates and adopting two-sorted first-order interpretation structures. In Section 4 we present an axiomatization for the enriched logic 2-FOL+AE, shown in Section 5 to be strongly complete over the class of interpretation structures with supported measures. The notion of supported measure generalizes that of discrete measure.

In Section 2, besides presenting the language and the semantics of FOL+AE, we classify the proposed AE quantifier following the taxonomy in [8]. In Section 3, after presenting the language and the semantics of 2-FOL+AE, we introduce the notion of supported measure that will be crucial in the proof of completeness. The axiomatization presented in Section 4 includes axioms for dealing with the two-sorted FOL fragment, axioms for dealing with AE, axioms for the interplay between the two classical quantifiers and AE, and the axiom characterizing supported measures (SE), plus the usual rules *Modus Ponens* (MP), $\forall$-generalization ($\forall$Gen) and $\forall^1$-generalization ($\forall^1$Gen). The axioms for AE make clear the similarities (normality) and the differences (instantiation) between AE and $\forall$. We conclude Section 4 with the meta-theorem of deduction and by proving the independence of some axioms. In Section 5 we prove the strong completeness of the axiomatization using a suitable revamp of the Lindenbaum–Henkin technique [13]. The usual $\exists$-witnesses are enough to provide SE-witnesses (for the existential counterpart of AE). Furthermore, although AE-instantiation is weaker than $\forall$-instantiation, things work out thanks to the SE axiom. We conclude Section 5 with some obvious but important corollaries of the completeness theorem. In particular, if a 2-FOL+AE theory has a (supported) model then it has a discrete model, implying the downward Lowenheim–Skölem theorem. Further developments of 2-FOL+AE, namely towards security applications, are discussed in Section 6.

2 First-order language and semantics

In this section we present first-order logic (FOL) enriched with a *modulated* quantifier (in the sense of [8]) denoted AE, where the intended meaning of $\mathsf{AE}x\varphi$ is "for almost all x, φ holds". To this end, we enrich the notion of first-order structure by adding a measure space on the domain; intuitively, a formula $\mathsf{AE}x\varphi$ will be satisfied if the set of values in the domain that can be assigned to x whilst falsifying φ has zero measure. By duality we obtain a quantifier SE, where $\mathsf{SE}x\varphi$ is read "there exist significantly many x such that φ holds" and is satisfied if the set of values that can be assigned to x whilst making φ true has non-zero measure. We assume that the reader is familiar with the basics of measure theory (at the level of the initial chapters of a textbook on the subject, for instance [11]).

We begin by defining terms and formulas of the logic FOL+AE.

Definition 2.1 Assume a given first-order signature $\Sigma = \langle F, P\rangle$ and a countable set $X = \{x_i \mid i \in \mathbb{N}\}$ of variables. Terms are generated in the usual way from X and F. Formulas are built inductively applying elements of P to terms or by using (some) propositional connectives, first-order quantifiers or the modulated quantifier AE.

$$\varphi \quad = \quad p(\bar{t}) \mid \mathrm{ff} \mid \varphi \Rightarrow \varphi \mid \forall x \varphi \mid \mathsf{AE} x \varphi$$

The remaining propositional connectives and the existential quantifier are defined as abbreviations in the usual way. Furthermore, the quantifier SE is defined by abbreviation by $\mathsf{SE}x\varphi \equiv \neg\mathsf{AE}x\neg\varphi$.

It is convenient to introduce some notation that will be needed throughout the paper.

NOTATION 2.2 The notation $\mathsf{var}(t)$ and $\mathsf{var}(\varphi)$ refers to the variables that occur in a term t or in a formula φ. In the latter case, $\mathsf{var}(\varphi)$ includes not only variables that occur in terms in φ (free or bound) but also variables being quantified upon (e.g. the y in $\forall y\psi$).

For example, $\mathsf{var}(f(x,a)) = \{x\}$ and $\mathsf{var}(\mathsf{AE}xp(y,b)) = \{x,y\}$.

NOTATION 2.3 The notation $t \triangleright x : \varphi$ stands for "term t is free for variable x in formula φ", with the usual meaning in FOL – namely, that if x is replaced by t in φ then no variables in t become bound.

In particular, $y \triangleright x : \varphi$ holds for any variable y that does not occur in φ (although this condition is by no means necessary).

DEFINITION 2.4 An interpretation structure is a tuple $\mathfrak{M} = \langle D, [\![\cdot]\!], \mathcal{B}, \mu\rangle$ where:

- D is a non-empty set;
- $\langle D, [\![\cdot]\!]\rangle$ is a first-order interpretation structure, that is:
  - for each $f \in F_n$, $[\![f]\!] : D^n \to D$;
  - for each $p \in P_n$, $[\![p]\!] : D^n \to \{0,1\}$.
- $\langle D, \mathcal{B}, \mu\rangle$ is a measure space, that is:
  - $\mathcal{B}$ is a σ-algebra over D;
  - μ is a measure on $\mathcal{B}$.
- $\mu(D) \neq 0$.

DEFINITION 2.5 Satisfaction in a structure $\mathfrak{M}$ given a variable assignment ρ is defined in the usual way as for FOL, with the following extra clause[2]:

$$\mathfrak{M}\rho \Vdash \mathsf{AE}x\varphi \text{ if there is } B \in \mathcal{B} \text{ such that } \left(|\varphi|^x_{\mathfrak{M}\rho}\right)^{\mathsf{c}} \subseteq B \text{ and } \mu(B) = 0$$

where $|\varphi|^x_{\mathfrak{M}\rho}$ (the extent of φ relative to x in $\mathfrak{M}$ with assignment ρ) is defined by[3]

$$|\varphi|^x_{\mathfrak{M}\rho} = \{d \mid \mathfrak{M}\,\rho^x_d \Vdash \varphi\}.$$

Validity and entailment are defined as expected.

[2]As usual, $(A)^{\mathsf{c}}$ denotes the complement of A.

[3]Throughout this paper, ρ^x_d denotes the assignment that takes x to d and behaves as ρ elsewhere.

PROPOSITION 2.6 The logic FOL+AE is a conservative extension of FOL.

PROOF. Formulas that do not use the modulated quantifier are satisfied in a structure with a given assignment iff they are satisfied in the corresponding FOL structure (i.e. the structure obtained by forgetting the measure on the domain). Since any FOL structure can be made into a structure of FOL+AE by adding e.g. the counting measure on its domain, it follows that the valid FOL-formulas in the extended logic are precisely the valid formulas of FOL. $\square$

PROPOSITION 2.7 If $\langle D, \mathcal{B}, \mu \rangle$ is a complete measure space and $\mathfrak{M}\rho \Vdash \mathsf{AE}x\varphi$, then $\left(|\varphi|^x_{\mathfrak{M}\rho}\right)^{\mathsf{c}}$ is measurable with measure 0.

PROOF. In a complete measure space, any subset of a zero-measure set is itself a zero-measure set. $\square$

REMARK 2.8 In view of Proposition 2.7, we could instead *define* $\mathfrak{M}\rho \Vdash \mathsf{AE}x\varphi$ to hold if $\mu(\left(|\varphi|^x_{\mathfrak{M}\rho}\right)^{\mathsf{c}}) = 0$. However, besides requiring the measure space to be complete (a constraint that may not be desirable), this definition is not suitable to generalization in the sense we will now discuss. If we replace $\mu(B) = 0$ with $\mu(B) < \varepsilon$ for some previously fixed ε we obtain a different notion of "almost everywhere", which can be relevant in some contexts (e.g. when $\langle D, \mathcal{B}, \mu \rangle$ is a probability space, the meaning of $\mathsf{AE}x\varphi$ then becoming "except with negligible probability"). This alternative notion will be discussed in the concluding section.

Dealing with this more general notion is the reason for introducing the set B in the definition above: while it is true that any subset of a zero-measure set is measurable in a complete measure space, it is not true in general that $|\varphi|^x_{\mathfrak{M}\rho}$ is measurable even if we assume that $[\![f]\!]$ and $[\![p]\!]$ are measurable for all $f \in F$ and $p \in P$, as the following example shows.

EXAMPLE 2.9 Let $\Sigma = \langle F, P \rangle$ be a first-order signature with $F_n = \emptyset$ for all $n \in \mathbb{N}$, $P_2 = \{p\}$ and $P_n = \emptyset$ for $n \neq 2$. Let $\mathfrak{M}$ be a first-order structure for Σ with domain $\mathbb{R}$ endowed with the usual measure such that

$$[\![p]\!](x, y) = \begin{cases} 1 & \text{if } x \in U \text{ or } x \neq y \\ 0 & \text{otherwise} \end{cases}$$

where $U \subseteq \mathbb{R}$ is any non-measurable set. Notice that $[\![p]\!]$ is a measurable function: $[\![p]\!]^{-1}(0)$ is a zero-measure set (it is contained in the line $x = y$), hence $[\![p]\!]^{-1}(1)$ is also measurable, since the union of these is $\mathbb{R}^2$. However, regardless of ρ, $|\forall y(p(x, y))|^x_{\mathfrak{M}\rho} = U$ is not measurable by hypothesis.

The following proposition gives some examples of formulas that hold in all structures.

PROPOSITION 2.10 The following formulas are valid.

1. $(\forall x\varphi) \Rightarrow (\mathsf{AE}x\varphi)$

2. $(\mathsf{AE}x\varphi) \Rightarrow (\exists x\varphi)$

3. $(\mathsf{AE}x\varphi) \Rightarrow (\mathsf{AE}y[\varphi]^x_y)$ whenever $y \notin \mathsf{var}(\varphi)$

4. $\mathsf{AE}y((\mathsf{AE}x\varphi) \Rightarrow [\varphi]^x_y)$ whenever $y \triangleright x : \varphi$ and y does not occur free in φ

5. $(\forall x(\varphi \Rightarrow \psi)) \Rightarrow ((\mathsf{AE}x\varphi) \Rightarrow (\mathsf{AE}x\psi))$

6. $(\forall x(\varphi \Leftrightarrow \psi)) \Rightarrow ((\mathsf{AE}x\varphi) \Leftrightarrow (\mathsf{AE}x\psi))$

7. $(\mathsf{AE}x(\varphi \Rightarrow \psi)) \Rightarrow ((\mathsf{AE}x\varphi) \Rightarrow (\mathsf{AE}x\psi))$

8. $((\mathsf{AE}x\varphi) \wedge (\mathsf{AE}x\psi)) \Leftrightarrow \mathsf{AE}x(\varphi \wedge \psi)$

9. $\mathsf{AE}x\mathsf{tt}$

10. $(\mathsf{AE}x\varphi) \Rightarrow \neg(\mathsf{AE}x(\neg\varphi))$

11. $(\mathsf{AE}x\varphi) \Rightarrow (\mathsf{SE}x\varphi)$

12. $((\mathsf{AE}x\varphi) \wedge (\mathsf{AE}x\psi)) \Rightarrow \exists x(\varphi \wedge \psi)$

Proof. These properties are direct consequences of the properties of measure functions, as we show. Let $\mathfrak{M}$ be an interpretation structure and ρ be some assignment.

1. Suppose that $\mathfrak{M}\rho \Vdash \forall x\varphi$; then $\mathfrak{M}\rho^x_d \Vdash \varphi$ for any $d \in D$, hence $\left(|\varphi|^x_{\mathfrak{M}\rho}\right)^{\mathsf{c}} = \emptyset$ and $\mu(\emptyset) = 0$, and thus $\mathfrak{M}\rho \Vdash \mathsf{AE}x\varphi$.

2. If $\mathfrak{M}\rho \Vdash \mathsf{AE}x\varphi$, then $|\varphi|^x_{\mathfrak{M}\rho}$ cannot be empty (otherwise it would have zero measure while its complement would be contained in a zero-measure set, implying $\mu(D) = 0$). Therefore $\mathfrak{M}\rho \Vdash \exists x\varphi$.

3. From the hypothesis that y does not occur in φ it follows trivially that $|\varphi|^x_{\mathfrak{M}\rho} = |[\varphi]^x_y|^y_{\mathfrak{M}\rho}$.

4. Suppose that $\mathfrak{M}\rho \nVdash \mathsf{AE}y((\mathsf{AE}x\varphi) \Rightarrow [\varphi]^x_y)$, that y does not occur free in φ, and that $y \triangleright x : \varphi$. Then the set $\left(|((\mathsf{AE}x\varphi) \Rightarrow [\varphi]^x_y)|^y_{\mathfrak{M}\rho}\right)^{\mathsf{c}}$ is not contained in any set of measure zero, hence it cannot be empty. For any d in that set, $\mathfrak{M}\rho^y_d \Vdash \mathsf{AE}x\varphi$ and $\mathfrak{M}\rho^y_d \nVdash [\varphi]^x_y$; but then the hypotheses on y imply that $\mathfrak{M}\rho \Vdash \mathsf{AE}x\varphi$ and $\mathfrak{M}\rho^x_d \nVdash \varphi$. It follows that $\left(|((\mathsf{AE}x\varphi) \Rightarrow [\varphi]^x_y)|^y_{\mathfrak{M}\rho}\right)^{\mathsf{c}} = \left(|[\varphi]^x_y|^y_{\mathfrak{M}\rho}\right)^{\mathsf{c}} = \left(|\varphi|^x_{\mathfrak{M}\rho}\right)^{\mathsf{c}}$, taking advantage of the fact that y does not occur free in φ. But this set is contained in a set with measure zero (since $\mathfrak{M}\rho \Vdash \mathsf{AE}x\varphi$), contradiction.

5. If $\mathfrak{M}\rho \Vdash \forall x(\varphi \Rightarrow \psi)$, then $|\varphi|^x_{\mathfrak{M}\rho} \subseteq |\psi|^x_{\mathfrak{M}\rho}$. Suppose that $\mathfrak{M}\rho \Vdash \mathsf{AE}x\varphi$; then $\left(|\psi|^x_{\mathfrak{M}\rho}\right)^{\mathsf{c}} \subseteq \left(|\varphi|^x_{\mathfrak{M}\rho}\right)^{\mathsf{c}} \subseteq B$ for some measurable B with $\mu(B) = 0$, hence $\mathfrak{M}\rho \Vdash \mathsf{AE}x\psi$.

6. Applying the reasoning in the previous proof twice it follows that $\mathfrak{M}\rho \Vdash (\forall x(\varphi \Leftrightarrow \psi)) \Rightarrow ((\mathsf{AE}x\varphi) \Leftrightarrow (\mathsf{AE}x\psi))$.

7. Suppose that $\mathfrak{M}\rho \Vdash \mathsf{AE}x(\varphi \Rightarrow \psi)$ and $\mathfrak{M}\rho \Vdash \mathsf{AE}x\varphi$. Then

$$\begin{aligned}
\left(|\psi|^x_{\mathfrak{M}\rho}\right)^{\mathsf{c}} &= \{d \mid \mathfrak{M}\rho \nVdash [\psi]^x_d\} \\
&= \{d \mid \mathfrak{M}\rho \nVdash [\psi]^x_d \text{ and } \mathfrak{M}\rho \Vdash [\varphi]^x_d\} \cup \{d \mid \mathfrak{M}\rho \nVdash [\psi]^x_d \text{ and } \mathfrak{M}\rho \nVdash [\varphi]^x_d\} \\
&\subseteq \{d \mid \mathfrak{M}\rho \nVdash [\varphi \Rightarrow \psi]^x_d\} \cup \{d \mid \mathfrak{M}\rho \nVdash [\varphi]^x_d\} \\
&= \left(|\varphi \Rightarrow \psi|^x_{\mathfrak{M}\rho}\right)^{\mathsf{c}} \cup \left(|\varphi|^x_{\mathfrak{M}\rho}\right)^{\mathsf{c}}
\end{aligned}$$

and by hypothesis each of these two sets is contained in a set of measure zero. Since the union of zero-measure sets still has measure zero, it follows that $\mathfrak{M}\rho \Vdash \mathsf{AE}x\psi$.

8. Notice that $\left(|\varphi \wedge \psi|^x_{\mathfrak{M}\rho}\right)^\mathsf{c} = \left(|\varphi|^x_{\mathfrak{M}\rho} \cap |\psi|^x_{\mathfrak{M}\rho}\right)^\mathsf{c} = \left(|\varphi|^x_{\mathfrak{M}\rho}\right)^\mathsf{c} \cup \left(|\psi|^x_{\mathfrak{M}\rho}\right)^\mathsf{c}$. If $\left(|\varphi \wedge \psi|^x_{\mathfrak{M}\rho}\right)^\mathsf{c} \subseteq B$ then B contains both $\left(|\varphi|^x_{\mathfrak{M}\rho}\right)^\mathsf{c}$ and $\left(|\psi|^x_{\mathfrak{M}\rho}\right)^\mathsf{c}$, which proves the converse implication supposing $\mu(B) = 0$. For the direct implication just consider the intersection of two sets $B_\varphi \supseteq \left(|\varphi|^x_{\mathfrak{M}\rho}\right)^\mathsf{c}$ and $B_\psi \supseteq \left(|\psi|^x_{\mathfrak{M}\rho}\right)^\mathsf{c}$.

9. The set $\left(|\mathrm{tt}|^x_{\mathfrak{M}\rho}\right)^\mathsf{c} = \emptyset$ always has zero measure.

10. Suppose that $\mathfrak{M}\rho \Vdash \mathsf{AE}x\varphi$. Then $\left(|\varphi|^x_{\mathfrak{M}\rho}\right)^\mathsf{c} \subseteq B$ for some set B such that $\mu(B) = 0$. Then $(B)^\mathsf{c}$ is measurable and $\mu((B)^\mathsf{c}) = \mu(D) - \mu(B) = \mu(D) \neq 0$. But $\left(|\neg\varphi|^x_{\mathfrak{M}\rho}\right)^\mathsf{c} = \left(\left(|\varphi|^x_{\mathfrak{M}\rho}\right)^\mathsf{c}\right)^\mathsf{c} = |\varphi|^x_{\mathfrak{M}\rho}$, and any set containing this must contain $(B)^\mathsf{c}$, hence its measure must also be $\mu(D)$. Therefore $\mathfrak{M}\rho \nVdash \mathsf{AE}x\neg\varphi$, hence $\mathfrak{M}\rho \Vdash \neg(\mathsf{AE}(\neg\varphi))$.

11. By definition of SE, the previous formula is precisely $(\mathsf{AE}x\varphi) \Rightarrow (\mathsf{SE}x\varphi)$.

12. Suppose that $\mathfrak{M}\rho \Vdash ((\mathsf{AE}x\varphi) \wedge (\mathsf{AE}x\psi))$; then there are sets $B_\varphi \supseteq \left(|\varphi|^x_{\mathfrak{M}\rho}\right)^\mathsf{c}$ and $B_\psi \supseteq \left(|\psi|^x_{\mathfrak{M}\rho}\right)^\mathsf{c}$ with $\mu(B_\varphi) = \mu(B_\psi) = 0$. It follows that $\mu(B_\varphi \cup B_\psi) = 0$, hence its complementary has positive measure and is contained in $|\varphi \wedge \psi|^x_{\mathfrak{M}\rho}$, thus the latter is not empty, whence $\exists x(\varphi \wedge \psi)$ holds.

Notice that removing the requirement $\mu(D) \neq 0$ only affects the proofs of validity of 2, 10, 11 and 12. Conversely, if either of these formulas holds in a structure for any φ, then in that structure necessarily $\mu(D) \neq 0$ (just take $\varphi = \mathrm{ff}$). □

REMARK 2.11 The requirement that y not occur free in φ in formula $\mathsf{AE}y((\mathsf{AE}x\varphi) \Rightarrow [\varphi]^x_y)$ is essential, as the following example shows. Let φ to be $x \neq y$ and $\mathfrak{M} = \langle \mathbb{R}, [\![\cdot]\!], \mathcal{B}, \mu \rangle$ with $\langle \mathbb{R}, \mathcal{B}, \mu \rangle$ the usual measure on the real line and $[\![\neq]\!]$ inequality.

Given an arbitrary ρ, $\mathfrak{M}\rho \Vdash \mathsf{AE}x\varphi$, since $\left(|\varphi|^x_{\mathfrak{M}\rho}\right)^\mathsf{c} = \{\rho(y)\}$, which has zero measure. On the other hand, $\mathfrak{M}\rho \nVdash [\varphi]^x_y$, since $\rho(y) = \rho(y)$. Therefore $\mathfrak{M}\rho \nVdash (\mathsf{AE}x\varphi) \Rightarrow [\varphi]^x_y$. Since ρ is arbitrary, this implies that $|(\mathsf{AE}x\varphi) \Rightarrow [\varphi]^x_y|^y_{\mathfrak{M}} = \emptyset$, so $\mathfrak{M} \nVdash \mathsf{AE}y((\mathsf{AE}x\varphi) \Rightarrow [\varphi]^x_y)$, even though y is free for x in φ.

PROPOSITION 2.12 The following entailments hold.

1. $\varphi, \varphi \Rightarrow \psi \models \psi$
2. $\varphi \models \forall x\varphi$
3. $\varphi \models \mathsf{AE}x\varphi$

PROOF. The first two are immediate consequences of the fact that interpretation structures of FOL+AE are first-order structures. The third follows from the fact that, if $\mathfrak{M} \Vdash \varphi$, then $|\varphi|^x_{\mathfrak{M}\rho} = D$ for any ρ, hence $\left(|\varphi|^x_{\mathfrak{M}\rho}\right)^\mathsf{c} = \emptyset$, and this set has measure zero. Thus $\mathfrak{M}\rho \Vdash \mathsf{AE}x\varphi$, and arbitrariness of ρ proves that $\mathfrak{M} \Vdash \mathsf{AE}x\varphi$. □

The authors of [8] classify quantifiers in several categories. According to Proposition 2.10, the quantifier AE is:

- a modulated quantifier, since it satisfies 1, 2, 6 and 3;
- a "most" quantifier, since it satisfies 5, 10 and 2;
- a "ubiquity" quantifier, consequence of 8 and 5.

Interestingly, AE is not an "almost all" quantifier in their sense, since such a quantifier ∇ must satisfy $(\nabla x\varphi) \vee (\nabla x \neg\varphi)$. This corresponds in our setting to the semantical requirements $\mu(|\varphi|^x_{\mathfrak{M}\rho}) = 0$ or $\mu\big((|\varphi|^x_{\mathfrak{M}\rho})^{\mathsf{c}}\big) = 0$. One can easily see that this is not necessarily valid by taking φ to be $p(x)$ in a structure where $D = \mathbb{N}$, $[\![p]\!](n) = 1$ iff n is even and μ is the counting measure on the natural numbers. A way of getting AE to behave in such a way is to follow the alternative definition suggested in Remark 2.8 taking $\varepsilon > 1/2$ and $\langle D, \mathcal{B}, \mu\rangle$ a probability space (so $\mu(D) = 1$). On the other hand, property 7 of the same Proposition states that AE as defined is a *normal* quantifier, so many of the previous properties are consequences of this fact (as will be shown in more detail in Section 4).

We conclude this section with a significant result.

PROPOSITION 2.13 The logic FOL+AE does not satisfy the downward Lowenheim–Skölem theorem.

PROOF. Without loss of generality, assume that $=$ denotes equality and let φ be the formula $\forall x(\mathsf{AE} y \neg(x = y))$, intuitively representing the semantic condition "singleton sets have measure zero". Clearly φ is satisfiable, since the usual measure on the real line has this property. However, it has no countable models: if $\mathfrak{M} = \langle D, [\![\cdot]\!], \mathcal{B}, \mu\rangle$ is a model of φ and D is countable, then for any assignment ρ we have that

$$D = \bigcup_{d\in D} \{d\} \subseteq \bigcup_{d\in D} |x = y|^y_{\mathfrak{M}\rho^x_d},$$

hence D is included in a countable union of sets of measure zero (since by hypothesis $\mathfrak{M}\rho^x_d \Vdash x = y$ for each d) and must be a zero-measure set itself.

Now observe that the only property of equality used above was reflexivity. The reasoning above works just as well if we take φ to be $(\forall x(\mathsf{AE} y \neg p(x, y))) \wedge (\forall x(p(x, x)))$ and assume nothing at all about the interpretation of p. $\square$

This result indicates that the usual (Henkin-style) completeness techniques for FOL cannot be applied to FOL+AE, since they always yield the downward Lowenheim–Skölem theorem as a corollary.

With this in mind, we considered a restricted class of interpretation structures (those with supported measures), which in turn required the availability of quantification over unary predicates when we came to the axiomatization stage. This second conservative extension of FOL is presented in the next section.

3 Extending the language

The language and the semantics of 2-FOL+AE are those of FOL+AE plus a (generalized) second-order quantifier.

DEFINITION 3.1 The formulas of 2-FOL+ÆE over a given first-order signature are generated by the following grammar.

$$\varphi \quad = \quad p(\bar{t}) \mid r(t) \mid \mathrm{ff} \mid \varphi \Rightarrow \varphi \mid \forall x \varphi \mid \text{Æ} x \varphi \mid \forall^1 r \varphi$$

Here, r stands for the unary predicate variables. As before, the remaining propositional connectives and the existential quantifiers $\exists$ and SE are defined by abbreviation; likewise, we abbreviate $\neg(\forall^1 r(\neg\varphi))$ to $\exists^1 r\varphi$.

Notice that we now have two kinds of variables. Henceforth, by *closed* we will mean closed for both. When we refer to a formula with one free first-order variable we will implicitly assume that no second-order variables are free in the formula, and likewise for formulas with one free second-order variable.

As mentioned before, we need to consider structures with measure functions satisfying some extra properties.

DEFINITION 3.2 A measure space $\langle D, \mathcal{B}, \mu\rangle$ is *discrete* if there are countable sets $\{d_i \mid i \in \mathbb{N}\} \subseteq D$ and $\{\omega_i \mid i \in \mathbb{N}\} \subseteq \mathbb{R}^+$ such that[4] $\mu(A) = \sum_{d_i \in A} \omega_i$ for any $A \in \mathcal{B}$.

Discrete probability spaces are examples of discrete measure spaces. Another example is the counting measure on any countable set. A less obvious example is the measure space $\langle \mathbb{R}, \mathcal{B}, \mu\rangle$ where a set is measurable iff it is the union of intervals $[n, n+1]$ with $n \in \mathbb{Z}$ and μ is the restriction to $\mathcal{B}$ of the usual Lebesgue measure. It is easy to see that $\mu(A) = |\{n \mid n \in \mathbb{Z} \text{ and } n + 1/2 \in A\}|$ (so $\omega_i = 1$ for all i).

DEFINITION 3.3 A measure space $\langle D, \mathcal{B}, \mu\rangle$ is *supported* if arbitrary unions of zero-measure sets are contained in a zero-measure set.

When the measure space is supported, we have: (i) there is a largest zero-measure set Z; (ii) for any set $A \in \mathcal{B}$, $\mu(A) = \mu(A \setminus Z)$. It can be shown that all discrete measure spaces are supported; the reverse implication does not hold, however: the counting measure is always supported, but it is discrete iff the domain is countable.

DEFINITION 3.4 An interpretation structure for 2-FOL+ÆE is a tuple $\langle D, D^1, [\![\cdot]\!], \mathcal{B}, \mu\rangle$ such that:

1. $\langle D, [\![\cdot]\!], \mathcal{B}, \mu\rangle$ is an interpretation structure for FOL+ÆE;
2. $\langle D, \mathcal{B}, \mu\rangle$ is a supported measure space;
3. $D^1 \subseteq \wp(D)$ containing the extents of all formulas with a single free first-order variable.

Assignments now take first-order variables to elements of D and second-order variables to elements of D^1. Satisfaction of formulas is defined inductively as before, with the following extra clauses for the second-order variables and quantifier.

$$\begin{array}{rcl} \mathfrak{M}\rho \Vdash r(t) & \text{iff} & [\![t]\!]_{\mathfrak{M}\rho} \in \rho(r) \\ \mathfrak{M}\rho \Vdash \forall^1 r\varphi & \text{iff} & \mathfrak{M}\rho^r_B \Vdash \varphi \text{ for any } B \in D^1 \end{array}$$

[4]The ω_is are seen as *weights* associated to the x_is.

Observe that we do not require any relationship between D^1 and $\mathcal{B}$ because it is not needed. Note also that $\forall^1$ is endowed with a Henkin-style generalized second-order semantics. Therefore, 2-FOL+AE is equivalent to two-sorted first-order logic plus AE, which justifies its name.

Remark 3.5 Since structures of 2-FOL+AE are enriched structures of monadic second-order logic, we have: $\varphi \models \forall^1 r\varphi$.

Proposition 3.6 The logic 2-FOL+AE is a conservative extension of FOL.

Proof. Analogous to Proposition 2.6. $\square$

Observe that 2-FOL+AE is not a conservative extension of FOL+AE since the former assumes that the measures are supported.

4 Axiomatization

In this section we define a Hilbert calculus for 2-FOL+AE. This calculus is sound, as Theorem 4.2 shows; in Section 5 we will show that it is also complete w.r.t. the supported-measure semantics given above.

Definition 4.1 The axiom system for 2-FOL+AE contains the following axioms.

Taut All instances of propositional tautologies.

K$\forall$ $(\forall x(\varphi \Rightarrow \psi)) \Rightarrow ((\forall x\varphi) \Rightarrow (\forall x\psi))$

I$\forall$ $(\forall x\varphi) \Rightarrow [\varphi]^x_t$ whenever $t \triangleright x : \varphi$

IAE $\mathsf{AE}y((\mathsf{AE}x\varphi) \Rightarrow [\varphi]^x_y)$ whenever $y \triangleright x : \varphi$ and y is not free in φ

K$\forall^1$ $(\forall^1 r(\varphi \Rightarrow \psi)) \Rightarrow ((\forall^1 r\varphi) \Rightarrow (\forall^1 r\psi))$

I$\forall^1$ $(\forall^1 r\varphi) \Rightarrow [\varphi]^r_\psi$ whenever ψ is a formula with a single first-order free variable and $\psi \triangleright r : \varphi$

Comp $\exists^1 r(\forall x(r(x) \Leftrightarrow \varphi))$ whenever φ is a formula with a single first-order free variable x and r is not free in φ

SE $(\mathsf{SE}x\varphi) \Rightarrow \exists x(\varphi \wedge \forall^1 r((\mathsf{AE}y(r(y))) \Rightarrow r(x)))$

The inference rules are generalization for the universal quantifiers ($\forall$Gen) and ($\forall^1$Gen) plus Modus Ponens (MP).

Some comments are in order at this stage. Axioms Taut, K$\forall$ (normality) and I$\forall$ (instantiation) are as in FOL. Indeed, the usual FOL axiom

$$\text{K}\forall' \quad (\forall x(\varphi \Rightarrow \psi)) \Rightarrow (\varphi \Rightarrow (\forall x\psi)) \text{ if } x \text{ does not occur free in } \varphi$$

and K$\forall$ above are inter-derivable in the presence of I$\forall$.

- In FOL we can derive K∀...

1.	$(\forall x(\varphi \Rightarrow \psi)) \Rightarrow (\varphi \Rightarrow (\forall x\psi))$	K∀′
2.	$\forall x(\varphi \Rightarrow \psi)$	Hyp
3.	$\varphi \Rightarrow (\forall x\psi)$	MP $1,2$
4.	$\forall x\varphi$	Hyp
5.	$(\forall x\varphi) \Rightarrow \varphi$	I∀
6.	φ	MP $5,4$
7.	$\forall x\psi$	MP $3,6$

- ... and in FOL without K∀′ we can derive it from K∀.

1.	$(\forall x(\varphi \Rightarrow \psi)) \Rightarrow (\forall x\varphi \Rightarrow \forall x\psi)$	K∀
2.	$\forall x(\varphi \Rightarrow \psi)$	Hyp
3.	$\forall x\varphi \Rightarrow \forall x\psi$	MP $1,2$
4.	φ	Hyp
5.	$\forall x\varphi$	∀Gen 4
6.	$\forall x\psi$	MP $3,5$

In both cases we use the Deduction Theorem for FOL.

We adopted K∀ instead of K∀′ because we want to make as clear as possible the similarities and the differences between ∀ and Æ: if we replace ∀ by Æ, the two resulting formulas

[KÆ] $(\text{Æ}x(\varphi \Rightarrow \psi)) \Rightarrow ((\text{Æ}x\varphi) \Rightarrow (\text{Æ}x\psi))$
[KÆ′] $(\text{Æ}x(\varphi \Rightarrow \psi)) \Rightarrow (\varphi \Rightarrow (\text{Æ}x\psi))$ where x does not occur free in φ

are *not* inter-derivable, because Æ does not enjoy full instantiation; only the second of the above derivations remains valid (so normality is stronger). Also, axiom K∀ is simpler since it makes no requirements on φ.

Formulas KÆ and IÆ are counterparts to K∀ and I∀. The latter was taken as an axiom, while the former is derivable as will be shown at the end of this section. Note that IÆ is a much weaker form of instantiation, reflecting the weaker quantification made by Æ. This fact is the source of the impossibility of deriving KÆ from KÆ′. In Proposition 4.9 we will show that generalization for the modulated quantifier can be derived and does not need to be added as an inference rule.

Axioms $\text{K}\forall^1$ and $\text{I}\forall^1$ should pose no questions after the discussion above, while axiom Comp is simply the unary second-order comprehension scheme.

Axiom SE states that, whenever φ holds significantly, there is a single point where it holds that is contained in no set of measure zero. This is equivalent to the semantic requirement that the measure be supported, as we show below. It also provides a restricted instantiation scheme for Æ comparable to I∀. Also note that the *interplay formulas*

[∀Æ] $(\forall x\varphi) \Rightarrow (\text{Æ}x\varphi)$
[Æ∃] $(\text{Æ}x\varphi) \Rightarrow (\exists x\varphi)$

are easily derivable from SE.

Soundness and axiom independence results

THEOREM 4.2 *(Soundness of 2-FOL+*AE*)* Let $\Gamma \cup \{\varphi\}$ be a set of formulas. If $\Gamma \vdash \varphi$ then $\Gamma \models \varphi$.

PROOF. By soundness of FOL, since all structures are first-order structures every instance of Taut, K$\forall$ and I$\forall$ is valid; by Proposition 2.10, all instances of axiom IAE are valid as well. Furthermore, since structures of 2-FOL+AE are enriched structures of monadic second-order logic, axioms K$\forall^1$, I$\forall^1$ and Comp hold.

The crucial step is to check the soundness of axiom SE. Assume that for some formula φ there exist a structure $\mathfrak{M}$ and an assignment ρ such that $\mathfrak{M}\rho \Vdash \mathsf{SE}x\varphi$ and $\mathfrak{M}\rho \nVdash \exists x(\varphi \wedge \forall^1 r((\mathsf{AE}y(r(y))) \Rightarrow r(x)))$. From the latter it follows that, for any $d \in |\varphi|^x_{\mathfrak{M}\rho}$, there exists a set $X_d \in D^1$ such that $\mu({X_d}^c) = 0$ and $d \notin X_d$. But then

$$|\varphi|^x_{\mathfrak{M}\rho} \subseteq \bigcup_{d \in |\varphi|^x_{\mathfrak{M}\rho}} {X_d}^c,$$

and hence $\mu(|\varphi|^x_{\mathfrak{M}\rho}) = 0$ (since $\mu({X_d}^c) = 0$ for all d, the union of all these sets is still contained in a zero-measure set by the fact that μ is supported), from which follows that $\mathfrak{M}\rho \Vdash \mathsf{AE}x\neg\varphi$. This contradicts $\mathfrak{M}\rho \Vdash \mathsf{SE}x\varphi$, hence the existence of such an $\mathfrak{M}$ and ρ is absurd. This shows that axiom SE is sound.

Finally, Proposition 2.12 and Remark 3.5 guarantee that the inference rules are sound. □

Observe that we obtain a seemingly incomplete but still useful sound calculus for FOL+AE by dropping the axioms and rules about $\forall^1$ and replacing axiom SE by KAE, $\forall\mathsf{AE}$ and $\mathsf{AE}\exists$.

PROPOSITION 4.3 *(Soundness within FOL+*AE*)* The calculus composed of axioms Taut, K$\forall$, I$\forall$, KAE, IAE, $\forall\mathsf{AE}$ and $\mathsf{AE}\exists$ plus inference rules MP and $\forall$Gen is sound with respect to the class of FOL+AE interpretation structures.

PROOF. Analogous to the previous proof, observing that the soundness of the FOL+AE components of the calculus does not depend on the measures being supported. □

PROPOSITION 4.4 *(Independence of* $\mathsf{AE}\exists$ *within FOL+*AE*)* Axiom $\mathsf{AE}\exists$ is not derivable from the remaining FOL+AE axioms.

PROOF. As discussed in the proof of Proposition 2.10, this axiom is equivalent to the property $\mu(D) \neq 0$ in the definition of structure for FOL+AE (Definition 2.5). If this requirement is removed all other axioms and inference rules remain sound w.r.t. the (larger) class of structures, which in turn does not satisfy $\mathsf{AE}\exists$. Hence this axiom is independent from the others. □

PROPOSITION 4.5 *(Independence of* KAE *within FOL+*AE*)* Axiom KAE is not derivable from the remaining FOL+AE axioms.

Proof. Replacing AE everywhere by $\exists$ in the calculus yields valid FOL formulas except in the case of KAE, since $(\exists x(\varphi \Rightarrow \psi)) \Rightarrow ((\exists x\varphi) \Rightarrow (\exists x\psi))$ does not hold, as is easily seen by taking ψ to be ff. This means that replacing AE by $\exists$ in any formula that can be derived in FOL+AE without using axiom KAE yields a valid FOL formula. Since this does not hold for KAE itself, this axiom cannot be derived from the others. $\square$

Observe that Propositions 4.4 and 4.5 still hold if we enrich FOL+AE with the unary second-order semantic features and adopt the usual axioms K$\forall^1$, I$\forall^1$ and Comp. Therefore, we can establish the following result.

Proposition 4.6 *(Independence of* SE *within 2-FOL+*AE*)* Axiom SE is not derivable from the remaining axioms.

Proof. Within 2-FOL+AE we can infer $\mathsf{AE}\exists$ and KAE from SE, as mentioned above. $\square$

Meta-theorems and rule admissibility

Let $\varphi_1, \ldots, \varphi_n$ be a derivation from a set of hypothesis Γ. Recall that φ_i is said to *depend* from the hypothesis $\gamma \in \Gamma$ if: either φ_i is γ; or φ_i is obtained by applying generalization to φ_j, which depends on γ; or φ_i is obtained by applying MP to φ_j and φ_k, and at least one of these depends on γ.

An application of generalization to φ in a derivation is said to be an *essential generalization over a dependent of* γ if φ depends on γ and the variable being generalized occurs free in γ.

Proposition 4.7 *(Deduction Theorem for 2-FOL+*AE*)* Let Γ be a set of formulas and φ, ψ be formulas. Suppose that $\Gamma \cup \{\varphi\} \vdash \psi$ and that in the derivation of ψ no essential generalizations were made over dependents of φ. Then $\Gamma \vdash \varphi \Rightarrow \psi$.

Proof. The proof of the Deduction Theorem for FOL applies here, since no new inference rules were added. $\square$

Corollary 4.8 Let Γ be a set of formulas and φ, ψ be formulas with φ closed. If $\Gamma \cup \{\varphi\} \vdash \psi$, then $\Gamma \vdash \varphi \Rightarrow \psi$.

Proof. If φ is closed, no essential generalizations over dependents of φ are possible, hence the Deduction Theorem applies. $\square$

We now turn our attention the rule concerning the introduction of the AE quantifier.

Proposition 4.9 *(Admissibility of* AEGen *within 2-FOL+*AE*)* The following rule of generalization for the almost-everywhere quantifier is admissible.

$$(\mathsf{AE}\text{Gen}) \quad \text{from } \varphi \text{ infer } \mathsf{AE}x\varphi$$

PROOF. Suppose that $\varphi_1, \dots, \varphi_n$ is a derivation where φ occurs at step n. Then we can proceed as follows.

$n.$	φ	
$n+1.$	$\forall x \varphi$	$\forall$Gen n
$n+2.$	$(\forall x \varphi) \Rightarrow (\mathsf{AE} x \varphi)$	$\forall\mathsf{AE}$
$n+3.$	$\mathsf{AE} x \varphi$	MP $n+2, n+1$

□

From this point onwards, we will use AEGen whenever helpful. Notice that, in applying the Deduction Theorem, care must be taken to verify that no essential generalizations over dependents of the hypothesis are implicitly made through the use of AEGen.

Useful theorems and alternative axiomatizations

As mentioned before, KAE is derivable in 2-FOL+AE. Consider the following derivation:

1.	$\forall^1 r((\mathsf{AE} y(r(y))) \Rightarrow r(x))$	Hyp
2.	$(\forall^1 r((\mathsf{AE} y(r(y))) \Rightarrow r(x))) \Rightarrow ((\mathsf{AE} y(\varphi \Rightarrow \psi)) \Rightarrow (\varphi^y_x \Rightarrow \psi^y_x))$	I$\forall^1$
3.	$(\mathsf{AE} y(\varphi \Rightarrow \psi)) \Rightarrow (\varphi^y_x \Rightarrow \psi^y_x)$	MP $1, 2$
4.	$\mathsf{AE} y(\varphi \Rightarrow \psi)$	Hyp
5.	$\varphi^y_x \Rightarrow \psi^y_x$	MP $3, 4$
6.	$(\forall^1 r(\mathsf{AE} y(r(y))) \Rightarrow r(x)) \Rightarrow ((\mathsf{AE} y \varphi) \Rightarrow \varphi^y_x)$	I$\forall^1$
7.	$(\mathsf{AE} y \varphi) \Rightarrow \varphi^y_x$	MP $1, 6$
8.	$\mathsf{AE} y \varphi$	Hyp
9.	φ^y_x	MP $7, 8$
10.	ψ^y_x	MP $5, 9$

By the Deduction Theorem we conclude that $\{\mathsf{AE} y(\varphi \Rightarrow \psi), \mathsf{AE} y \varphi\} \vdash (\forall^1 r(\mathsf{AE} y(r(y))) \Rightarrow r(x)) \Rightarrow \psi^y_x$. Notice that axiom SE can be rewritten equivalently as

$$[\mathsf{SE}'] \quad (\forall x((\forall^1 r((\mathsf{AE} y(r(y))) \Rightarrow r(x))) \Rightarrow \varphi)) \Rightarrow (\mathsf{AE} x \varphi)$$

using de Morgan laws. We proceed towards KAE as follows:

1.	$(\forall^1 r(\mathsf{AE} y(r(y))) \Rightarrow r(x)) \Rightarrow \psi^y_x$	Hyp
2.	$\forall x((\forall^1 r(\mathsf{AE} y(r(y))) \Rightarrow r(x)) \Rightarrow \psi^y_x)$	$\forall$Gen 1
3.	$(\forall x((\forall^1 r((\mathsf{AE} y(r(y))) \Rightarrow r(x))) \Rightarrow \psi^y_x)) \Rightarrow (\mathsf{AE} x \psi)$	SE'
4.	$\mathsf{AE} x \psi^y_x$	MP $2, 3$

Finally, by applying MP twice and using axiom IAE we obtain KAE.

The interplay between $\forall$ and AE can be axiomatized in different ways within FOL+AE. An interesting possibility is replacing $\mathsf{AE}\exists$ by the following formula.

$$(\mathsf{AESE}) \quad (\mathsf{AE} x \varphi) \Rightarrow (\mathsf{SE} x \varphi)$$

This formula is a counterpart to the FOL theorem $(\forall x \varphi) \Rightarrow (\exists x \varphi)$. It is easily derivable within FOL+AE, recalling that negation and significant existence are defined by abbreviation. The first lemma we use in the following derivation will be proved in the next proposition (its proof does not require $\mathsf{AE}\exists$), while the second one is a simple FOL theorem.

1.	$\mathsf{AE}x\varphi$	Hyp
2.	$\mathsf{AE}x(\neg\varphi)$	Hyp
3.	$((\mathsf{AE}x\varphi) \wedge (\mathsf{AE}x(\neg\varphi))) \Rightarrow \mathsf{AE}x(\varphi \wedge (\neg\varphi))$	Lemma
4.	$(\mathsf{AE}x\varphi) \Rightarrow ((\mathsf{AE}x(\neg\varphi)) \Rightarrow ((\mathsf{AE}x\varphi) \wedge (\mathsf{AE}x(\neg\varphi))))$	Taut
5.	$(\mathsf{AE}x(\neg\varphi)) \Rightarrow ((\mathsf{AE}x\varphi) \wedge (\mathsf{AE}x(\neg\varphi)))$	MP $4, 1$
6.	$(\mathsf{AE}x\varphi) \wedge (\mathsf{AE}x(\neg\varphi))$	MP $5, 2$
7.	$\mathsf{AE}x(\varphi \wedge (\neg\varphi))$	MP $3, 6$
8.	$(\mathsf{AE}x(\varphi \wedge (\neg\varphi))) \Rightarrow (\exists x(\varphi \wedge (\neg\varphi)))$	$\mathsf{AE}\exists$
9.	$\exists x(\varphi \wedge (\neg\varphi))$	MP $8, 7$
10.	$(\exists x(\varphi \wedge (\neg\varphi))) \Rightarrow \mathrm{ff}$	Lemma
11.	ff	MP $10, 9$

Applying the Deduction Theorem twice yields the conclusion.

Conversely, from AESE we can derive $\mathsf{AE}\exists$.

1.	$\mathsf{AE}x\varphi$	Hyp
2.	$(\mathsf{AE}x\varphi) \Rightarrow \neg(\mathsf{AE}x(\neg\varphi))$	AESE
3.	$\neg(\mathsf{AE}x(\neg\varphi))$	MP $2, 1$
4.	$(\forall x(\neg\varphi)) \Rightarrow (\mathsf{AE}x(\neg\varphi))$	$\forall\mathsf{AE}$
5.	$((\forall x(\neg\varphi)) \Rightarrow (\mathsf{AE}x(\neg\varphi))) \Rightarrow ((\neg\mathsf{AE}x(\neg\varphi)) \Rightarrow (\neg\forall x(\neg\varphi)))$	Taut
6.	$(\neg\mathsf{AE}x(\neg\varphi)) \Rightarrow \neg\forall x(\neg\varphi)$	MP $5, 4$
7.	$\neg\forall x(\neg\varphi)$	MP $6, 3$

The last formula abbreviates to $\exists x\varphi$; the Deduction Theorem establishes $\mathsf{AE}\exists$.

Proposition 4.10 All the statements in Proposition 2.10 are derivable in FOL+AE. Furthermore, the following dependencies hold.

- 8 requires KAE and $\forall\mathsf{AE}$;
- 10, 11 and 12 require $\mathsf{AE}\exists$ and 6 (and hence also KAE and $\forall\mathsf{AE}$).

Proof. We first show that all formulas are derivable. The numbering is the same as in Proposition 2.10.

1., 2., 4. and 7. These formulas are all axioms.

3. The second step is to show that $(\mathsf{AE}x\varphi) \Rightarrow (\mathsf{AE}y[\varphi]^x_y)$ holds whenever y does not occur in φ. We invoke the Deduction Theorem.

1.	$\mathsf{AE}x\varphi$	Hyp
2.	$\mathsf{AE}y((\mathsf{AE}x\varphi) \Rightarrow [\varphi]^x_y)$	IAE
3.	$(\mathsf{AE}y((\mathsf{AE}x\varphi) \Rightarrow [\varphi]^x_y)) \Rightarrow ((\mathsf{AE}y\mathsf{AE}x\varphi) \Rightarrow (\mathsf{AE}y[\varphi]^x_y))$	KAE
4.	$(\mathsf{AE}y\mathsf{AE}x\varphi) \Rightarrow (\mathsf{AE}y[\varphi]^x_y)$	MP $3, 2$
5.	$\mathsf{AE}y\mathsf{AE}x\varphi$	AEGen 1
6.	$\mathsf{AE}y[\varphi]^x_y$	MP $4, 5$

Notice that the Deduction Theorem applies since the generalization in step 5 is over y, which is not free in $\mathsf{AE}x\varphi$.

5. For $(\forall x(\varphi \Rightarrow \psi)) \Rightarrow ((\mathsf{AE}x\varphi) \Rightarrow (\mathsf{AE}x\psi))$ we again invoke the Deduction Theorem.

1.	$\forall x(\varphi \Rightarrow \psi)$	Hyp
2.	$(\forall x(\varphi \Rightarrow \psi)) \Rightarrow (\mathsf{AE}x(\varphi \Rightarrow \psi))$	$\forall\mathsf{AE}$
3.	$\mathsf{AE}x(\varphi \Rightarrow \psi)$	MP $2, 1$
4.	$(\mathsf{AE}x(\varphi \Rightarrow \psi)) \Rightarrow ((\mathsf{AE}x\varphi) \Rightarrow (\mathsf{AE}x\psi))$	KAE
5.	$(\mathsf{AE}x\varphi) \Rightarrow (\mathsf{AE}x\psi)$	MP $4, 3$

6. We now show $(\forall x(\varphi \Leftrightarrow \psi)) \Rightarrow ((\mathsf{AE}x\varphi) \Leftrightarrow (\mathsf{AE}x\psi))$. Let z stand for a variable not occurring in either φ or ψ, so that $z \triangleright x : \varphi$ and $z \triangleright x : \psi$.

1.	$(\varphi \Leftrightarrow \psi) \Rightarrow (\varphi \Rightarrow \psi)$	Taut
2.	$\forall x((\varphi \Leftrightarrow \psi) \Rightarrow (\varphi \Rightarrow \psi))$	$\forall$Gen 1
3.	$(\forall x((\varphi \Leftrightarrow \psi) \Rightarrow (\varphi \Rightarrow \psi))) \Rightarrow ((\forall x(\varphi \Leftrightarrow \psi)) \Rightarrow (\forall x(\varphi \Rightarrow \psi)))$	K$\forall$
4.	$(\forall x(\varphi \Leftrightarrow \psi)) \Rightarrow (\forall x(\varphi \Rightarrow \psi))$	MP $3, 2$
5.	$\forall x(\varphi \Leftrightarrow \psi)$	Hyp
6.	$\forall x(\varphi \Rightarrow \psi)$	MP $4, 5$
7.	$(\forall x(\varphi \Rightarrow \psi)) \Rightarrow (\mathsf{AE}x(\varphi \Rightarrow \psi))$	$\forall\mathsf{AE}$
8.	$\mathsf{AE}x(\varphi \Rightarrow \psi)$	MP $7, 6$
9.	$(\mathsf{AE}x(\varphi \Rightarrow \psi)) \Rightarrow ((\mathsf{AE}x\varphi) \Rightarrow (\mathsf{AE}x\psi))$	KAE
10.	$(\mathsf{AE}x\varphi) \Rightarrow (\mathsf{AE}x\psi)$	MP $9, 8$

Repeating this proof with φ and ψ interchanged (except in the premisse of 1) we obtain the converse implication, from which the bi-implication follows by propositional reasoning; the desired formula is then a consequence of the Deduction Theorem.

8. The proof of $((\mathsf{AE}x\varphi) \wedge (\mathsf{AE}x\psi)) \Leftrightarrow \mathsf{AE}x(\varphi \wedge \psi)$ is done in several steps. We first show each of the implications separately, using the Deduction Theorem, and then invoke propositional reasoning to show the bi-implication.

The right-to-left implication is proved as follows.

1.	$\mathsf{AE}x(\varphi \wedge \psi)$	Hyp
2.	$\varphi \wedge \psi \Rightarrow \varphi$	Taut
3.	$\mathsf{AE}x(\varphi \wedge \psi \Rightarrow \varphi)$	AEGen 2
4.	$(\mathsf{AE}x(\varphi \wedge \psi \Rightarrow \varphi)) \Rightarrow ((\mathsf{AE}x(\varphi \wedge \psi)) \Rightarrow (\mathsf{AE}x\varphi))$	KAE
5.	$(\mathsf{AE}x(\varphi \wedge \psi)) \Rightarrow (\mathsf{AE}x\varphi)$	MP $4, 3$
6.	$\mathsf{AE}x\varphi$	MP $5, 1$

Reasoning in a similar way we derive $\mathsf{AE}x\psi$, whence we get by propositional reasoning and the Deduction Theorem that $(\mathsf{AE}x(\varphi \wedge \psi)) \Rightarrow ((\mathsf{AE}x\varphi) \wedge (\mathsf{AE}x\psi))$.

For the converse we proceed as follows.

1.	$\mathsf{AE}x\varphi$	Hyp
2.	$\mathsf{AE}x\psi$	Hyp
3.	$(\mathsf{AE}x(\varphi \Rightarrow (\psi \Rightarrow (\varphi \wedge \psi)))) \Rightarrow ((\mathsf{AE}x\varphi) \Rightarrow (\mathsf{AE}x(\psi \Rightarrow (\varphi \wedge \psi))))$	KAE
4.	$\varphi \Rightarrow (\psi \Rightarrow (\varphi \wedge \psi))$	Taut
5.	$\mathsf{AE}x(\varphi \Rightarrow (\psi \Rightarrow (\varphi \wedge \psi)))$	AEGen 4
6.	$(\mathsf{AE}x\varphi) \Rightarrow (\mathsf{AE}x(\psi \Rightarrow (\varphi \wedge \psi)))$	MP $3, 5$
7.	$\mathsf{AE}x(\psi \Rightarrow (\varphi \wedge \psi))$	MP $6, 1$
8.	$(\mathsf{AE}x(\psi \Rightarrow (\varphi \wedge \psi))) \Rightarrow ((\mathsf{AE}x\psi) \Rightarrow (\mathsf{AE}x(\varphi \wedge \psi)))$	KAE
9.	$(\mathsf{AE}x\psi) \Rightarrow (\mathsf{AE}x(\varphi \wedge \psi))$	MP $8, 7$
10.	$\mathsf{AE}x(\varphi \wedge \psi)$	MP $9, 2$

By applying the Deduction Theorem we obtain $((\mathsf{AE}x\varphi) \wedge (\mathsf{AE}x\psi)) \Rightarrow (\mathsf{AE}x(\varphi \wedge \psi))$, from which the conclusion again follows by propositional reasoning.

9. The proof that $\mathsf{AE}x\mathsf{tt}$ is derivable is straightforward.

1.	tt	Taut
2.	$\mathsf{AE}x\mathsf{tt}$	AEGen 1

10. and 11. These two formulas are the same since SE is defined as an abbreviation; they were shown above to be derivable.

12. Finally, we show that $((\mathsf{AE}x\varphi) \wedge (\mathsf{AE}x\psi)) \Rightarrow \exists x(\varphi \wedge \psi)$. We use as lemma a formula already derived in the proof of (6).

1.	$(\mathsf{AE}x\varphi) \wedge (\mathsf{AE}x\psi)$	Hyp
2.	$((\mathsf{AE}x\varphi) \wedge (\mathsf{AE}x\psi)) \Rightarrow \mathsf{AE}x(\varphi \wedge \psi)$	Lemma
3.	$\mathsf{AE}x(\varphi \wedge \psi)$	MP $2, 1$
4.	$(\mathsf{AE}x(\varphi \wedge \psi)) \Rightarrow (\exists x(\varphi \wedge \psi))$	$\mathsf{AE}\exists$
5.	$\exists x(\varphi \wedge \psi)$	MP $4, 3$

The result once again follows from the Deduction Theorem.

As for the dependencies, using the last three formulas and the remaining axioms one can easily derive $\mathsf{AE}\exists$ (part of this fact was shown above), so the dependency is really essential.

Also, according to the proof of Proposition 4.5, it is enough to verify that $((\exists x\varphi) \wedge (\exists x\psi)) \Leftrightarrow \exists x(\varphi \wedge \psi)$ is not a theorem of FOL (which it is not) to establish that $((\mathsf{AE}x\varphi) \wedge (\mathsf{AE}x\psi)) \Leftrightarrow \mathsf{AE}x(\varphi \wedge \psi)$ cannot be proved without using axiom KAE, so this dependency is also essential. $\square$

5 Completeness

The completeness proof for 2-FOL+AE follows the structure of the usual completeness proof for FOL: we reduce the problem to showing that any consistent set of closed formulas has a model and focus on constructing a *term model* for a given set of closed formulas whose domain is the set of closed terms over a defined extension of the language. First we show that any consistent set of formulas has a maximal consistent extension, using the usual Lindenbaum construction. Afterwards, we add existential (Henkin) witnesses for formulas of the form $\neg\forall x\varphi$ (equivalent to $\exists x\neg\varphi$) and $\neg\forall^1 r\varphi$ (equivalent to $\exists^1 r\neg\varphi$) while preserving consistency. From this extended signature we build a term model, to which we assign a measure function by looking at the syntactic extent of formulas.

Definition 5.1 A set Γ is said to be *consistent* if there is a formula φ such that $\Gamma \nvdash \varphi$.

Lemma 5.2 Suppose φ is closed. If $\Gamma \nvdash \neg\varphi$ then $\Gamma \cup \{\varphi\}$ is consistent.

Proof. Assume that $\Gamma \cup \{\varphi\}$ is inconsistent; then $\Gamma \cup \{\varphi\} \vdash \psi$ for any formula ψ, hence in particular $\Gamma \cup \{\varphi\} \vdash \neg\varphi$. Since φ is closed, the corollary to the Deduction Theorem applies and we conclude that $\Gamma \vdash \varphi \Rightarrow \neg\varphi$. But $\Gamma \vdash (\varphi \Rightarrow \neg\varphi) \Rightarrow \neg\varphi$, since the latter formula is an instance of a propositional tautology. By MP it follows that $\Gamma \vdash \neg\varphi$, from which our lemma follows by counter-reciprocal. $\square$

This result allows us to prove completeness in the following way. To show that if $\Gamma \models \varphi$ then $\Gamma \vdash \varphi$, we assume that φ is closed and that $\Gamma \nvdash \varphi$; by the previous lemma, $\Gamma \cup \{\neg\varphi\}$ is consistent. Then we will build a model for $\Gamma \cup \{\neg\varphi\}$, contradicting the assumption that $\Gamma \models \varphi$. If φ is not closed we simply take its universal closure $\forall\varphi$.

Definition 5.3 A set Γ is said to be *maximal consistent* if it is consistent and, for every closed formula φ, either $\varphi \in \Gamma$ or $\Gamma \cup \{\varphi\}$ is inconsistent.

Definition 5.4 A set Γ is *exhaustive* if it is consistent and, for every closed formula φ, either $\varphi \in \Gamma$ or $\neg\varphi \in \Gamma$.

Lemma 5.5 A set Γ is maximal consistent iff it is exhaustive.

Proof. If Γ is not consistent the result is trivial, so suppose Γ is consistent.

Assume Γ is exhaustive. Then Γ is maximal consistent: given ψ closed, either $\psi \in \Gamma$ or $\neg\psi \in \Gamma$, and in the latter case $\Gamma \cup \{\psi\}$ is inconsistent.

Assume Γ is not exhaustive, and suppose without loss of generality that it is deductively closed (if it were not closed, then any $\psi \in (\Gamma^\vdash \setminus \Gamma)$ would contradict maximality of Γ). Then there is some closed formula φ such that $\varphi \notin \Gamma$ and $\neg\varphi \notin \Gamma$; equivalently, since Γ is closed, $\varphi \notin \Gamma$ and $\Gamma \nvdash \neg\varphi$. By Lemma 5.2, $\Gamma \cup \{\varphi\}$ is a consistent extension of Γ, hence Γ is not maximal consistent. $\square$

Proposition 5.6 Suppose Γ is consistent. Then there is an exhaustive extension of Γ, which we will denote by $\overline{\Gamma}$.

Proof. Let $\varphi_0, \ldots, \varphi_n, \ldots$ be an enumeration of the closed formulas over Σ and consider the following sequence of sets of formulas.

$$\begin{aligned} \Gamma_0 &= \Gamma \\ \Gamma_{n+1} &= \begin{cases} (\Gamma_n \cup \{\varphi_n\})^\vdash & \text{if } \Gamma_n \nvdash \neg\varphi_n \\ \Gamma_n & \text{otherwise} \end{cases} \end{aligned}$$

By Lemma 5.2, induction proves that each Γ_n is consistent. Take their union $\overline{\Gamma} = \cup_{n\in\mathbb{N}}\Gamma_n$. Then:

- $\overline{\Gamma}$ is consistent: otherwise there is some closed φ for which $\varphi \in \overline{\Gamma}$ and $\neg\varphi \in \overline{\Gamma}$, whence by definition of $\overline{\Gamma}$ there are i and j for which $\varphi \in \Gamma_i$ and $\neg\varphi \in \Gamma_j$, and then $\Gamma_{\max(i,j)}$ would be inconsistent;
- $\overline{\Gamma}$ is exhaustive: we already showed that $\overline{\Gamma}$ is consistent; furthermore, any closed ψ is φ_n for some n, so either $\Gamma_n \not\vdash \neg\psi$, from which $\psi \in \Gamma_{n+1}$ and therefore $\psi \in \overline{\Gamma}$, or $\Gamma_n \vdash \neg\psi$, from which follows (since Γ_n is closed) that $\neg\psi \in \Gamma_n$ and therefore $\neg\psi \in \overline{\Gamma}$.

□

From this point onwards we fix a signature Σ^0. Let $\{c_n \mid n \in \mathbb{N}\}$ be a set of constants such that no c_n occurs in Σ^0, $\{p_n \mid n \in \mathbb{N}\}$ be a set of unary predicate symbols with the same property, and denote by Σ^+ the signature obtained by adding the c_ns and the p_ns to Σ^0. Let $\{\psi_n^+ \mid n \in \mathbb{N}\}$ be an enumeration of the formulas over Σ^+ with one free first-order variable and $\{\theta_n^+ \mid n \in \mathbb{N}\}$ be an enumeration of the formulas over Σ^+ with one free second-order variable. Let y_n stand for the free variable in formula ψ_n^+ and s_n for the free variable in formula θ_n^+. Let Γ^0 be consistent over Σ^0.

Lemma 5.7 Let γ_n and δ_n denote the following formulas, for each $n \in \mathbb{N}$.

$$\begin{aligned}\gamma_n &= (\neg(\forall y_n \psi_n^+)) \Rightarrow \neg[\psi_n^+]_{c_n}^{y_n}\\ \delta_n &= (\neg(\forall^1 s_n \theta_n^+)) \Rightarrow \neg[\theta_n^+]_{p_n}^{s_n}\end{aligned}$$

Consider the following sequence of sets of formulas.

$$\begin{aligned}\Gamma'_0 &= \Gamma^0\\ \Gamma'_{2n+1} &= (\Gamma'_{2n} \cup \{\gamma_n\})^{\vdash}\\ \Gamma'_{2n+2} &= \left(\Gamma'_{2n+1} \cup \{\delta_n\}\right)^{\vdash}\end{aligned}$$

Then $\Gamma' = \cup_{n\in\mathbb{N}}\Gamma'_n$ is consistent.

Proof. Suppose that Γ' is not consistent. Then there is some n for which Γ'_n is not consistent; consider now the minimal such n. There are two cases to consider.

(i) If $n = 0$, then Γ^0 is inconsistent, which is absurd: the usual proof for FOL that consistent sets over a signature are consistent over a larger signature can be applied in this setting.

(ii) Take now $n > 0$. The proof is very similar according to whether n is even or odd, so suppose without loss of generality that $n = 2k+1$. Then $\Gamma'_{2k} \cup \{\gamma_k\} \vdash \neg\gamma_k$. Since $(\gamma_k \Rightarrow \neg\gamma_k) \Rightarrow \neg\gamma_k$ is an instance of a propositional tautology and γ_k is closed, the corollary to the Deduction Theorem and propositional reasoning imply that $\Gamma'_{2k} \vdash \neg\gamma_k$. Hence we conclude that $\Gamma'_{2k} \vdash \neg\forall y_k \psi_k^+$ and $\Gamma'_{2k} \vdash [\psi_k^+]_{c_k}^{y_k}$. By induction on the length of the derivation of $[\psi_k^+]_{c_k}^{y_k}$ it is easy to check that $\Gamma'_{2k} \vdash [\psi_k^+]_{z}^{y_k}$, where z is some fresh variable not appearing in the original derivation. Applying generalization and α-equivalence for $\forall$ (which is a (meta-)theorem in FOL) we conclude that $\Gamma'_{2k} \vdash \forall y_k \psi_k^+$, so Γ'_{2k} is also inconsistent. This contradicts the assumption that n was the minimal n for which Γ'_n was inconsistent.

If $n = 2k + 2$ the reasoning is analogous replacing γ_k^+ by θ_k^+, y_k by s_k and c_k by p_k everywhere.

□

By the last result and Proposition 5.6, there is an exhaustive extension of Γ', which is also an exhaustive extension of Γ^0 w.r.t. the signature Σ^+. We denote this extension $\overline{\Gamma'}$ by Γ^+. We use Γ^+ to build a canonical model for Γ^0 in a way that deviates little from the standard first-order techniques.

DEFINITION 5.8 Let Γ^+ be an exhaustive set of formulas. The set $\mathcal{H}_{\Gamma^+}$ is the set $\{t \mid [\psi_n^+]_t^{y_n} \in \Gamma^+ \text{ whenever } (\text{Æ} y_n \psi_n^+) \in \Gamma^+\}$.

In other words, $\mathcal{H}_{\Gamma^+}$ is the set of terms that are relevant from the point of view of Æ ("*heavy*" terms). This set will be relevant to define a measure on the canonical model.

DEFINITION 5.9 The structure $\mathfrak{M}^+ = \langle D, [\![\cdot]\!]^+, \mathcal{B}, \mu\rangle$ is defined as follows.

- D is the set of closed Σ^+-terms.
- D^1 contains all sets of the form $\{t \mid p(t) \in \Gamma^+\}$ for some predicate symbol p in Σ^+.
- The interpretation of any constant or function symbol is itself.
- For any values $d_1, \ldots, d_n \in D$, $[\![p(d_1, \ldots, d_n)]\!]^+$ holds if $p(d_1, \ldots, d_n) \in \Gamma^+$.
- $\mathcal{B} = \wp(D)$.
- For $A \subseteq D$, $\mu(A)$ is defined as the number of heavy terms in A, that is, $\mu(A) = |A \cap \mathcal{H}_{\Gamma^+}|$.

The structure $\mathfrak{M}^0 = \langle D, [\![\cdot]\!]^0, \mathcal{B}, \mu\rangle$ is obtained by taking $[\![c]\!]^0 = [\![c]\!]^+$, $[\![f]\!]^0 = [\![f]\!]^+$ and $[\![p]\!]^0 = [\![p]\!]^+$ for constants c, function symbols f and predicate symbols p in Σ^0. Notice that $\mathfrak{M}^0$ is an interpretation structure for Σ^0.

It is straightforward to check that $\mathfrak{M}^+$ and $\mathfrak{M}^0$ are well-defined structures. In particular, μ is a supported measure.

PROPOSITION 5.10 Let φ^+ be a closed formula over Σ^+. Then $\mathfrak{M}^+ \Vdash \varphi^+$ iff $\varphi^+ \in \Gamma^+$.

PROOF. First, observe that a simple proof by structural induction shows that $[\![t]\!]^+ = t$ for any closed term t. We now prove the thesis by induction on the structure of closed formula φ^+.

If φ^+ is $p(t_1, \ldots, t_n)$ or $r(d)$, then the thesis holds by definition of $\mathfrak{M}^+$.

If φ^+ is $\neg\psi^+$, then $\mathfrak{M}^+ \Vdash \varphi^+$ iff $\mathfrak{M}^+ \nVdash \psi^+$ (by definition of satisfaction) iff $\psi^+ \notin \Gamma^+$ (by induction hypothesis) iff $\neg\psi^+ \in \Gamma^+$ (since Γ^+ is exhaustive).

If φ^+ is $\psi^+ \Rightarrow \gamma^+$, then $\mathfrak{M}^+ \Vdash \varphi^+$ iff (1) $\mathfrak{M}^+ \nVdash \psi^+$ or (2) $\mathfrak{M}^+ \Vdash \gamma^+$. If (1) holds then $\psi^+ \notin \Gamma^+$ (by induction hypothesis) hence $\neg\psi^+ \in \Gamma^+$ (since Γ^+ is exhaustive) and thus $\psi^+ \Rightarrow \gamma^+ \in \Gamma^+$ (since Γ^+ is closed). If (2) holds then $\gamma^+ \in \Gamma^+$ (by induction hypothesis) and again $\psi^+ \Rightarrow \gamma^+ \in \Gamma^+$ (since Γ^+ is closed). If neither (1) nor (2) holds then $\psi^+ \in \Gamma^+$ and $\gamma^+ \notin \Gamma^+$ (by induction hypothesis) hence $\neg\gamma^+ \in \Gamma^+$ (since Γ^+ is exhaustive) and thus $\neg(\psi^+ \Rightarrow \gamma^+) \in \Gamma^+$ (since Γ^+ is closed) whence $\psi^+ \Rightarrow \gamma^+ \notin \Gamma^+$ (since Γ^+ is consistent).

If φ^+ is $\forall x\psi^+$ then there are two cases. If ψ^+ is itself closed the result follows trivially from the induction hypothesis. Otherwise, ψ^+ has one free variable and hence φ^+ is (α-equivalent to) $\forall y_n\psi_n^+$ for some n. There are two cases to consider.

- Suppose that $\mathfrak{M}^+ \nVdash \forall y_n\psi_n^+$. Then $\mathfrak{M}^+ \nVdash [\psi_n^+]_d^{y_n}$ for some $d \in D$. By definition of D, d must be a closed term over Σ^+, so by induction hypothesis $[\psi_n^+]_d^{y_n} \notin \Gamma^+$. By exhaustiveness of Γ^+ it follows that $\neg[\psi_n^+]_d^{y_n} \in \Gamma^+$ and therefore $\Gamma^+ \vdash \neg[\psi_n^+]_d^{y_n}$; but $\Gamma^+ \vdash (\forall y_n\psi_n^+) \Rightarrow [\psi_n^+]_d^{y_n}$, hence by propositional reasoning it follows that $\Gamma^+ \vdash \neg(\forall y_n\psi_n^+)$. Since Γ^+ is consistent we conclude that $(\forall y_n\psi_n^+) \notin \Gamma^+$.

- Suppose now that $\forall y_n\psi_n^+ \notin \Gamma^+$. By exhaustiveness of Γ^+, it follows that $\neg(\forall y_n\psi_n^+) \in \Gamma^+$. By construction, $(\neg(\forall y_n\psi_n^+) \Rightarrow \neg[\psi_n^+]_{c_n}^{y_n}) \in \Gamma^+$, hence by MP we conclude that $\neg[\psi_n^+]_{c_n}^{y_n} \in \Gamma^+$. But Γ^+ is consistent, hence $[\psi_n^+]_{c_n}^{y_n} \notin \Gamma^+$ and therefore $\mathfrak{M}^+ \nVdash [\psi_n^+]_{c_n}^{y_n}$ by induction hypothesis, hence $\mathfrak{M}^+ \nVdash \forall y_n\psi_n^+$.

The case when φ^+ is $\forall^1 x\psi^+$ is analogous to the previous.

Finally suppose that φ^+ is $\mathsf{AE}x\psi^+$. Again the case where ψ^+ is closed follows trivially from the induction hypothesis. Otherwise, ψ^+ has one free variable and hence φ^+ is again (α-equivalent to) $\mathsf{AE}y_n\psi_n^+$ for some n, using axiom IAE. There are two cases to consider.

- Suppose that $\mathfrak{M}^+ \nVdash \mathsf{AE}y_n\psi_n^+$. Then $\left(|\psi_n^+|_{\mathfrak{M}^+}^{y_n}\right)^{\mathsf{c}} \subseteq B$ implies $\mu(B) > 0$. Since in this structure all sets are measurable, this implies that in particular $\mu(\left(|\psi_n^+|_{\mathfrak{M}^+}^{y_n}\right)^{\mathsf{c}}) > 0$, hence there is some heavy term t for which $\mathfrak{M}^+ \nVdash [\psi_n^+]_t^{y_n}$. By induction hypothesis $[\psi_n^+]_t^{y_n} \notin \Gamma^+$. By exhaustiveness of Γ^+ it follows that $\neg[\psi_n^+]_t^{y_n} \in \Gamma^+$. But by definition of heavy term this implies that $(\mathsf{AE}y_n\psi_n^+) \notin \Gamma^+$.

- Suppose now that $\mathsf{AE}y_n\psi_n^+ \notin \Gamma^+$. By exhaustiveness of Γ^+, it follows that $\neg(\mathsf{AE}y_n\psi_n^+)$ is in Γ^+ and, therefore, so is $(\mathsf{SE}y_n\neg\psi_n^+)$. By axiom SE and exhaustiveness, also $\exists y_n((\neg\psi_n^+) \wedge \forall^1 r((\mathsf{AE}y(r(y))) \Rightarrow r(y_n))) \in \Gamma^+$. Since the formula inside the existential quantifier has one free first-order variable, it must be ψ_k for some k, and hence we conclude that $[(\neg\psi_n^+) \wedge \forall^1 r((\mathsf{AE}y(r(y))) \Rightarrow r(y_n))]_{c_k}^{y_n} \in \Gamma^+$, whence from exhaustiveness $[\neg\psi_n^+]_{c_k}^{y_n} \in \Gamma^+$ and $[\forall^1 r((\mathsf{AE}y(r(y))) \Rightarrow r(y_n))]_{c_k}^{y_n} \in \Gamma^+$. By induction hypothesis $\mathfrak{M}^+ \Vdash [\neg\psi_n^+]_{c_k}^{y_n}$; again by exhaustiveness, if $\mathsf{AE}y_j\psi_j^+ \in \Gamma^+$ then also $[\psi_j^+]_{c_k}^{y_j} \in \Gamma^+$, hence c_k is heavy. Then $\mu(\{c_k\}) = 1$ and $\{c_k\} \subseteq \left(|\psi_n^+|_{\mathfrak{M}^+}^{y_n}\right)^{\mathsf{c}}$, hence by monotonicity of measures we conclude that $\mathfrak{M}^+ \not\models \mathsf{AE}y_n\psi_n^+$.

This conclude the proof. $\square$

Corollary 5.11 Let φ^0 be a closed formula over Σ^0. Then $\mathfrak{M}^0 \Vdash \varphi^0$ iff $\varphi^0 \in \Gamma^0$.

Proof. A proof by induction on the construction of Γ^+ shows that, for φ^0 over Σ^0, it is the case that $\varphi^0 \in \Gamma^0$ iff $\varphi^0 \in \Gamma^+$, since Γ^0 is maximal consistent over Σ^0. By the previous proposition, the latter is equivalent to $\mathfrak{M}^+ \Vdash \varphi^0$. A simple proof by induction again shows that this happens iff $\mathfrak{M}^0 \Vdash \varphi^0$. □

PROPOSITION 5.12 Γ^0 has a model.

PROOF. By Corollary 5.11 the canonical model $\mathfrak{M}^0$ (Definition 5.9) is a model of Γ^0. □

Furthermore, we know a lot about the nature of this model.

PROPOSITION 5.13

1. Γ^0 has a model with a countable domain.
2. Γ^0 has a discrete model.

PROOF. The model constructed above has a countable domain, and hence a discrete measure. □

The construction shown above leads to a model with a counting measure. Thus, since the set of heavy constants may be denumerable, the measure of the domain can be infinite. However, it is straightforward to adapt the construction in order to get a finite measure: enumerating $\mathcal{H}_{\Gamma^+}$ and assigning $\mu(t_k) = 1/2^{k+1}$ will yield a probability measure if this set is infinite.

THEOREM 5.14 *(Completeness)*

1. The deductive system for 2-FOL+Æ is complete w.r.t. the class of supported interpretation structures.
2. The deductive system for 2-FOL+Æ is complete w.r.t. the class of discrete interpretation structures.

PROOF.

1. The proof is by counter-reciprocal. Let Γ be a set of formulas and φ be a formula, and suppose that $\Gamma \nvdash \varphi$. Then $\Gamma \nvdash \forall\varphi$, where $\forall\varphi$ denotes the universal closure of φ. By Lemma 5.2, $\Gamma \cup \{\neg\forall\varphi\}$ is consistent. By Proposition 5.12 there is a model of $\Gamma \cup \{\neg\forall\varphi\}$; in particular, it is a model of Γ that does not satisfy $\forall\varphi$ and therefore neither does it satisfy φ. Hence $\Gamma \not\models \varphi$.
2. Analogous using Proposition 5.13.

□

COROLLARY 5.15 *(Compactness)* The logic 2-FOL+Æ is compact, i.e. if $\Gamma \models \varphi$ then there is a finite subset $\Psi \subseteq \Gamma$ such that $\Psi \models \varphi$.

Proof. Assume that $\Gamma \models \varphi$. By completeness it follows that $\Gamma \vdash \varphi$. Since derivations are finite, in any given derivation of φ from Γ only a finite number of formulas in Γ can be used. Pick a derivation, and take Ψ to be the set of these formulas. Then $\Psi \vdash \varphi$, and by soundness $\Psi \models \varphi$. $\square$

Corollary 5.16 *(Semi-decidability)* The logics FOL+AE and 2-FOL+AE are both semi-decidable, that is, the set of valid formulas is recursively enumerable but not recursive.

Proof. In both logics, the set of all derivations is recursively enumerable, since the set of sequences of formulas is recursively enumerable and deciding whether a given sequence is a valid derivation is recursive. This immediately yields a recursive enumeration of the set of valid formulas: they are the last formulas in valid derivations.

On the other hand, if this set were recursive then FOL would be decidable, since both logics have been shown to be conservative extensions of FOL (Propositions 2.6 and 3.6): given a FOL formula it would be enough to check whether it were valid in FOL+AE or 2-FOL+AE. Since FOL is not decidable, neither can any of the latter be. $\square$

6 Concluding remarks

Motivated by current concerns in the logics of security, we enriched FOL with a measure-theoretic "for almost all" quantifier AE. This quantifier turned out to be, according to the taxonomy in [8], a modulated quantifier, a "most" quantifier, and a "ubiquity" quantifier, but, interestingly, not an "almost all" quantifier. Nevertheless, we feel justified to say that AE is an "almost everywhere" quantifier given its measure-theoretic semantics. We established a sound calculus for FOL+AE and argued that it could not be made complete. By slightly restricting the class of structures and adding restricted second-order quantification to the language, we defined a new logic 2-FOL+AE endowed with a complete axiomatization. The proof of completeness uses a revamped version of the Lindenbaum–Henkin technique.

Towards further development of the idea of enriching FOL towards a full-fledged kleistic logic for applications in security, we now consider some variants of 2-FOL+AE and discuss how their study might be pursued.

A very simple generalization is obtained by replacing in the definition of satisfaction the clause for $\mathfrak{M}\rho \Vdash \mathsf{AE}x\varphi$ by the following.

$$\mathfrak{M}\rho \Vdash \mathsf{AE}x\varphi \text{ if there is } B \in \mathcal{B} \text{ such that } \left(|\varphi|^x_{\mathfrak{M}\rho}\right)^{\mathsf{c}} \subseteq B \text{ and } \mu(B) < \varepsilon$$

(In measure theory, this is sometimes referred to as "the interior measure of $\left(|\varphi|^x_{\mathfrak{M}\rho}\right)^{\mathsf{c}}$ is at least ε".)

The motivation for this can be seen as relaxing the condition for a set (of values that do not satisfy a given formula) to be considered insignificant. Instead of requiring that it have zero measure, we only insist that its measure be smaller than a given quantity ε (but the logic remains qualitative).

Unfortunately, this small change makes the resulting logic non-normal, since the class of sets whose measure is bounded by ε is no longer necessarily closed under union. Furthermore, if the total measure of the domain is finite (for example, if $\langle D, \mathcal{B}, \mu\rangle$ is a probability space) other properties like $(\mathsf{AE}x\varphi) \vee (\mathsf{AE}x\neg\varphi)$ may hold instead.

In the case where no restrictions are placed on $\mu(D)$ other than it be positive, there is hope that a complete axiomatization can be found for which a similar proof technique will establish completeness. Unfortunately, if $\mu(D)$ is finite the technique itself is not *a priori* applicable: there will be no way to have more than $\lfloor \mu(D)/\varepsilon \rfloor$ significant existential witnesses in the canonical model, since they form disjoint measurable sets; and it is easy to produce a sequence of formulas that requires an infinite number of existential witnesses from just one unary predicate symbol p as shown by the following sequence $\varphi_1, \dots, \varphi_n, \dots$, where $t_{i_1}, \dots, t_{i_n}, \dots$ are heavy terms in the canonical model and i_k is such that φ_k is $\neg(\mathsf{AE}x\psi^+_{i_k})$.

$$
\begin{aligned}
\varphi_1 &\equiv \mathsf{SE}xp(x) \\
\varphi_2 &\equiv \mathsf{SE}x(p(x) \wedge \neg p(t_{i_1})) \\
&\vdots \\
\varphi_{n+1} &\equiv \mathsf{SE}x(p(x) \wedge \neg p(t_{i_1}) \wedge \ldots \wedge \neg p(t_{i_n})) \\
&\vdots
\end{aligned}
$$

With the standard semantics, the set $\{\varphi_n \mid n \in \mathbb{N}\}$ is consistent, and its canonical model will require an infinite number of witnesses.

In this context, another generalization that arises naturally is allowing different modulated quantifiers to be interpreted by constraints involving different values of ε. The most interesting scenario is when $\mu(D)$ is finite; without loss of generality, we may suppose that $\mu(D) = 1$, so that $\langle D, \mathcal{B}, \mu\rangle$ is in fact a probability space. A possible setting that still keeps the language countable is to allow a countable set of modulated quantifiers AE_ε, with $\varepsilon \in \mathbb{Q}$, satisfying properties like the following.

$$
\begin{gathered}
(\mathsf{AE}_\varepsilon x\varphi) \Rightarrow (\mathsf{AE}_\delta x\varphi) \quad \text{if } \varepsilon \leq \delta \\
((\mathsf{AE}_\varepsilon x\varphi) \wedge (\mathsf{AE}_\delta x\varphi)) \Rightarrow (\mathsf{AE}_{\varepsilon+\delta} x\varphi) \\
\neg(\mathsf{SE}_{1+\varepsilon} x\varphi)
\end{gathered}
$$

For security applications, this line of research will lead naturally to a "securely everywhere" quantifier with the following intended meaning: $\mathsf{S}x\varphi$ holds iff the probability of an attacker falsifying φ by an appropriate choice of the value of x is negligible. The relationship between S and AE_ε would require an inference rule, given the implicit universal quantification over ε in one direction.

Notice that this variant yields a logic that includes quantitative features, yet still has a qualitative flavor and retains the usual FOL terms. The study of such a kleistic logic will be the object of future research.

In a different direction, it seems worthwhile to study the relationship between the proposed model-theoretic quantifiers and those based on topology-theoretic semantics, such as a "densely everywhere" quantifier or the "ubiquity" quantifier in [8].

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